MATH 521A: Abstract Algebra
Preparation for Exam 3

1. Prove that $f(x) = x^3 + 9x^2 + 8x + 12508477$ is irreducible in $\mathbb{Q}[x]$.
2. Prove that $f(x) = x^4 + x^3 + x^2 + x + 1$ is irreducible in $\mathbb{Q}[x]$.
3. Factor $f(x) = x^5 + 2x^3 + 2x^2 - x + 3$ into irreducibles in $\mathbb{Z}_5[x]$.
4. Factor $f(x) = x^5 + x^4 + 2x^3 + 2x^2 + 2x + 1$ into irreducibles in $\mathbb{Z}_5[x]$.
5. Set $f(x) = x^n - x^{n-2} \in F[x]$. Carefully find all divisors of $f(x)$ in $F[x]$.
6. Set $f(x) = x^5 + 3x^4 + 2x^3 + x^2 + 3x + 2, g(x) = x^4 + x^2 + 1$, both in $\mathbb{Z}_5[x]$. Use the extended Euclidean algorithm to find $\gcd(f, g)$ and to find polynomials $a(x), b(x)$ such that $\gcd(f(x), g(x)) = a(x)f(x) + b(x)g(x)$.
7. Let $f(x), g(x) \in R[x]$. Prove that $\deg(f + g) \leq \max\{\deg(f), \deg(g)\}$.
8. Let R be an integral domain. Prove that all linear polynomials in $R[x]$ are irreducible, if and only if R is a field.
9. Let $f(x), g(x), h(x) \in F[x]$. Suppose that $f(x)|g(x)h(x)$ and $\gcd(f(x), g(x)) = 1$. Prove that $f(x)|h(x)$.
10. Let p be an odd prime. Prove there is at least one $a \in \mathbb{Z}_p$ such that $x^2 - a$ is irreducible in $\mathbb{Z}_p[x]$.
11. If $f(x) \in F[x]$ is nonconstant and monic, prove that we may always write $f(x)$ as the product of irreducible monic polynomials.
12. We call a polynomial in $F[x]$ *cinom* if its constant coefficient is 1. If $f(x) \in F[x]$ is nonconstant and cinom, prove that we may always write $f(x)$ as the product of irreducible cinom polynomials.
13. Prove or disprove: If $f(x) \in F[x]$ is nonconstant, and both monic and cinom, then we may always write $f(x)$ as the product of irreducible polynomials that are both monic and cinom.
14. Let R be an integral domain. Let $f(x), g(x) \in R[x]$. Recall that we call f, g *associates* if there is some unit $u \in R$ such that $f(x) = ug(x)$. Prove that f, g are associates if and only if they divide each other (i.e. $f(x)|g(x)$ and $g(x)|f(x)$).
15. Let R be an integral domain. In $R[x]$, prove that "is an associate of" is an equivalence relation.